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SYSTEMS WITH CONTROL DELAYS

(NASA-CR-185802) REGULATOR PROBLEM FOR HEREDITARY DIFFERENTIAL SYSTEMS WITH CONTROL DELAYS (ICASE) 32 p

N89-71428

Unclas

00/63 0224369

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Report No. 82-3

February 10, 1982

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Operated by the

UNIVERSITIES SPACE RESEARCH ASSOCIATION

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ABSTRACT

In this paper we extend the result due to Vinter - Kwong (SIAM J. Control Optim., 19 (1981) pp. 139-153) on a regulator problem for a hereditary differential system with delays in the control to the case when point delays are included in the control. The main difference from the case when point delays are excluded is that we have to deal with an unbounded, unclosable input operator in our abstract formulation.

Research for this work was supported under NASA Contract No. NAS1-15810 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

1. Introduction

In this paper we consider a quadratic cost control problem both in finite and infinite time for a linear functional differential equation with delays in the control. The class of problems considered is fairly general and may include "point delays" in the control.

Such a control problem has been studied by a number of authors [4], [7], [8], and [11]. For the case when point delays are excluded in the control, complete results are available [11]. Reference [8] is concerned with the infinite time problem for the case when only point delays are present in the state and control. In that paper, however, no justification of existence and uniqueness of solutions to a steady state version of a Riccati-like equation which characterizes the optimal feedback operator was obtained.

The method used here is based upon an equivalence between the solution of functional differential equation with delays in the control and that of a "dual" evolution equation, which is motivated by the idea of Vinter-Kwong [11]. By using this equivalence, the quadratic cost control problem may be transformed into a dual control problem, which has been discussed in [5] as a dual deterministic control problem to the filtering problem for a stochastic functional differential equation with delays in the observation.

It is shown that the optimal control to the finite time problem may be expressed in a feedback form in terms of a unique solution to a Riccati equation which includes an unbounded operator in the quadratic term. It is necessary to overcome this unboundedness when the steady state solution of the Riccati equation is discussed. The principal result is that under the usual assumptions of stabilizability and detectability the optimal solution to the infinite time problem may be expressed in a feedback form through a unique solution of an "algebraic Riccati equation" and that the closed loop system is stable.

As an application of the above we may discuss the stability of a filter equation for a stochastic functional differential equation with delays in the observation.

The following notation will be used throughout the paper. $L_2(I;\mathbb{R}^\alpha)$ is the Hilbert space of $\mathbb{R}^\alpha$-valued, square integrable functions on the compact interval I. In the special case where $I = [-r,0]$ with $r > 0$ we shall abbreviate the notation and simply write L_2. In such a case the range space of the functions is determined by the context. $L^{loc}(I;\mathbb{R}^\alpha)$ is the Hilbert space of $\mathbb{R}^\alpha$-valued locally square integrable functions on the semi-infinite interval I.

If X and Y are Banach spaces, then the space of bounded operators from X into Y is denoted by $\mathcal{L}(X,Y)$. $\mathcal{D}(A)$ denotes the domain of an operator A. The adjoint of a densely defined linear operator A from one Hilbert space to another is written by A^*. The transpose of a matrix A is denoted by A^T. χ_I denotes the characteristic function of the set I.

We denote by M_2 the product space $\mathbb{R}^N \times L_2([-r,0];\mathbb{R}^N)$. Given an element $\phi \varepsilon M_2$, $\phi^0 \varepsilon \mathbb{R}^N$ and $\phi^1 \varepsilon L_2$ denote the two coordinates of ϕ, $\phi = (\phi^0,\phi^1)$. The symbol $\langle\langle \cdot,\cdot \rangle\rangle$ stands for the natural inner product in M_2. All the other inner products will be denoted by $\langle \cdot,\cdot \rangle$ and the underlying space is understood from the context.

$\|\cdot\|$ denotes the norm of elements of a Banach space and of operators between Banach spaces. Given a measurable function $x:[-r,\infty) \to \mathbb{R}^\alpha$ and $t > 0$, the function $x_t:[-r,0] \to \mathbb{R}^\alpha$ is defined by $x_t(\theta) = x(t+\theta)$.

2. The Functional Differential Equation

Consider the inhomogeneous functional integral equation in $\mathbb{R}^N$;

$$x(t) = \phi^0 + \int_0^t L\, x_s ds + \int_0^t Bu_s ds, \tag{2.1}$$

$$x_0 = \phi^1 \varepsilon L_2 \quad \text{and} \quad u_0 = \eta.$$

The control function $u(\cdot)$ takes values in $\mathbb{R}^m$ and $B : L_2([-r,0];\mathbb{R}^m) \to \mathbb{R}^N$ is defined by

$$Bu = B_0 u(0) + \int_{-r}^{0} d\zeta(\theta)u(\theta), \tag{2.2}$$

$$\zeta(\theta) = \sum_{i=1}^{k} B_i \chi_{[\theta_i,0]}(\theta) + \int_{-r}^{\theta} B(\xi)d\xi, \qquad \theta \varepsilon [-r,0],$$

where

$$-r \leq \theta_k \leq \cdots < \theta_1 < 0 \quad \text{and}$$

and

$$\sum_{i=1}^{k} B_i + \int_{-h}^{0} B(\theta)\, d\theta < \infty.$$

The linear map $L : L_2([-r,0] ; \mathbb{R}^N) \to \mathbb{R}^N$ is represented by

$$L\,\phi = \int_{-r}^{0} d\mu(\theta)\phi(\theta) + A_0\phi(0), \tag{2.3}$$

$$\mu(\theta) = \sum_{i=1}^{k} A_i \chi_{[\theta_i,0]} + \int_{-r}^{\theta} A(\xi)d\xi, \qquad \theta \varepsilon [-r,0],$$

where

$$\sum_{i=1}^{1} A_i + \int_{-r}^{0} A(\xi)\, d\xi < \infty.$$

Note that if $u \in L_2^{loc}([-r,0];\mathbb{R}^m)$ then the inhomogeneous term "Bu_t" is locally square integrable. Hence it follows from [3] that there exists a unique locally absolutely continuous solution to (2.1) with $\phi = (\phi^0,\phi^1) \in M_2$ and $u \in L_2^{loc}([-r,\infty): \mathbb{R}^m)$.

3. A Dual Evolution Equation

Let us consider the "transposed" homogeneous functional integral equation:

$$x(t) = \phi^0 + \int_0^t L^T x_s ds, \qquad (3.1)$$

$$x_0 = \phi^1 \in L_2,$$

where

$$L^T\phi = A_0^T\phi(0) + \int_{-r}^0 d\mu^T(\theta)\phi(\theta) \qquad \text{for} \quad \phi \in L_2([-r,0];\ \mathbb{R}^N).$$

From known results [3], [10] we may state the following.

<u>Lemma 3.1</u> Let the family $T(t), t \geq 0$ be the solution semi-group on M_2 associated with (3.1). Then

(i) its infinitesimal generator A is characterized by

$$\mathcal{D}(A) = \{\phi = (\phi^0,\phi^1) \in M_2 | \dot{\phi}^1 \in L_2 \text{ and } \phi^0 = \phi^1(0)\},$$

$$A\phi = (L^T\phi,\dot{\phi}), \qquad \text{for } \phi \in \mathcal{D}(A);$$

(ii) the adjoint A^* of A generates a strongly continuous semigroup $T^*(t)$, $t \geq 0$ on M_2 and is defined as follows:

$$\mathcal{D}(A^*) = \{\phi = (\phi^0,\phi^1) \in M_2 \mid \dot{z} \in L_2 \text{ and } z(-r) = 0\},$$

where

$$z(\theta) = \phi^1(\theta) - \int_{-r}^{\theta} d\mu(\xi)\phi^0, \qquad \theta \in [-r,0],$$

$$[A^*\phi]^0 = A_0\phi^0 + \phi^1(0),$$

$$[A^*\phi]^1(\theta) = -\dot{z}(\theta), \qquad \theta \in [-r,0]$$

Define the linear map $\mathcal{B}:M_2 \to \mathbb{R}^m$ by

$$\mathcal{B}\phi = B^T{}_0\phi^0 + \int_{-r}^{0} d\zeta^T(\theta)\phi^1(\theta) \quad \text{for} \quad \phi = (\phi^0,\phi^1) \in M_2. \tag{3.2}$$

Then $\mathcal{D}(\mathcal{B}) \supset \mathcal{D}(A)$.

$\mathcal{D}(A)$ itself is a Hilbert space when endowed with the inner product

$$\langle x,y\rangle_{\mathcal{D}(A)} = \langle\langle x,y\rangle\rangle + \langle\langle Ax,Ay\rangle\rangle \quad \text{for} \quad x,y \in \mathcal{D}(A), \tag{3.3}$$

and will be denoted by X. Recall the following ([6, Lemma 3.2 and 3.3]):

<u>LEMMA 3.2.</u> Let X' be the strong dual space of X with respect to M_2-norm. Then

(i) $T'(t)$, $t > 0$ defines a strongly continuous semigroup on X' and $\mathcal{D}(A') = M_2$.

(ii) $\mathcal{B}T(\cdot) \in \mathcal{L}(M_2, L_2([0,T];\ \mathbb{R}^m))$ for any $T > 0$ and

$$(\mathcal{B}T(\cdot))^*u = \int_0^T T'(s)\mathcal{B}'u(s)ds \quad \text{for} \quad u \in L_2([0,T];\ \mathbb{R}^m),$$

where (′) denotes the dual operators.

Note that

$$(3.4) \qquad \mathcal{B}'u = \left(B_0 u, \frac{d}{d\theta}\left(\int_{-r}^{\theta} d\zeta(\xi)u\right)\right) \varepsilon\ X' \quad \text{for} \quad u \ \varepsilon\ \mathbb{R}^m.$$

Let us consider a dual evolution equation in X';

$$(3.5) \qquad \frac{d}{dt}\tilde{x}(t) = \mathcal{A}'\,\tilde{x}(t) + \mathcal{B}'u(t),$$
$$\tilde{x}(0) = \phi\ \varepsilon\ X'.$$

Then (3.5) has a unique mild solution

$$(3.6) \qquad \tilde{x}(t) = T'(t)\phi + \int_0^t T'(t-s)\mathcal{B}'u(s)ds.$$

Since $T'(t)_{M_2}$, the restriction of $T'(t)$ to M_2 is equal to $T^*(t)$ for each $t \geq 0$, , by virtue of Lemma 3.2, $\tilde{x}(\cdot)$ is a continuous function in M_2 for $\phi\ \varepsilon\ M_2$ and $u\ \varepsilon\ L_2^{loc}([0,\infty);\ \mathbb{R}^m)$.

4. Equivalence

In this section we derive an equivalence between the solution to (2.1) and the function $\tilde{x}(\cdot)$ defined by (3.6). For this purpose, we introduce the continuous, linear map $M(\cdot,\cdot) : M_2 \times L_2([-r,0]\ ;\ \mathbb{R}^m) \to M_2$

$$(4.1) \qquad M\left((\phi^0,\phi^1),v(\cdot)\right) = \left(\phi^0,m(\cdot)\right),$$

where

$$m(\theta) = \int_{-r}^{\theta} d\mu(\xi)\phi^1(\xi-\theta) + \int_{-r}^{\theta} d\zeta(\xi)v(\xi-\theta).$$

We remark that our operator M is a natural extension of that introduced in [8] and [11].

Theorem 4.1. Suppose that $x(t)$, $t \geq -r$, is the solution to (2.1) and $\tilde{x}(t)$, $t \geq 0$ is defined by (3.6), with initial function $\psi = M(\phi,\eta)$. Then for $t \geq 0$

$$\tilde{x}(t) = M\big((x(t),\ x_t(\cdot)),\ u_t(\cdot)\big). \tag{4.2}$$

Moreover for $t \geq r$ and for arbitrary initial function ψ

$$\tilde{x}(t) = M\big((\tilde{x}^0(t), x_t^0(\cdot)), u_t(\cdot)\big), \tag{4.3}$$

where $\tilde{x}^0(t)$ is the first component of $\tilde{x}(t)$.

Proof: Assume $\psi \in \mathcal{D}(A^*)$ and $u \in U = \{$absolute continuous functions with $u(0) = 0$ and locally square integrable derivative $\}$. Then

$$A'\tilde{x}(t) = T^*(x)A^*\psi + \int_0^t T'(t-s)B'\dot{u}(s)ds - B'u(t) \quad \text{in} \quad X'.$$

Hence $A'\tilde{x}(t) + B'u(t)$ is continuous in the topology of M_2, which when combined with the fact that $\tilde{x}(\cdot)$ is a strong solution to (3.5), implies $\tilde{x}(t)$ is continuously differentiable in the topology of M_2.

Now $\tilde{x}(t)$ must satisfy

$$\frac{d}{dt}\tilde{x}^0(t) = A_0\tilde{x}^0(t) + \tilde{x}^1(t;0) + B_0u(t), \tag{4.4}$$

$$\frac{d}{dt}\tilde{x}^1(t;\theta) = -\frac{d}{d\theta}z(t;\theta) + \frac{d}{d\theta}\Big(\int_{-r}^{\theta} d\zeta(\xi)u(t)\Big), \tag{4.5}$$

where

$$z(t;\theta) = \tilde{x}^1(t;\theta) - \int_{-r}^{\theta} d\mu(\xi)\tilde{x}^0(t), \qquad \text{for } \theta \in [-r,0]. \tag{4.6}$$

Define for $t \geq 0$ and $\theta \in [-r,0]$

$$\tilde{z}(t;\theta) = z(t;\theta) - \int_{-r}^{\theta} d\zeta(\xi)u(t). \tag{4.7}$$

Then $\tilde{z}(t;\theta)$ is differentiable in θ and we write (4.5) as an equation for $\tilde{z}(t;\theta)$:

$$\frac{d}{dt}\tilde{z}(t;\theta) = -\frac{d}{d\theta}\tilde{z}(t;\theta) - \int_{-r}^{\theta} d\mu(\xi)\dot{\tilde{x}}^0(t) - \int_{-r}^{\theta} d\zeta(\xi)\dot{u}(t).$$

Now consider the semigroup of truncated right shifts on L_2, $\{\Phi(t),\ t \geq 0\}$

$$\big(\Phi(t)g(\cdot)\big)(\theta) = \begin{cases} g(\theta-t), & -r \leq \theta-t \leq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{4.8}$$

for $\theta \in [-r,0]$ and $g \in L_2$. Then $(-d/d\theta)(\cdot)$ is its infinitesimal generator and $\tilde{z}(0;\cdot)$ evolves in the domain $\{\phi \in L_2, \dot{\phi} \in L_2$ and $\phi(-r) = 0\}$.

We may view $\tilde{z}(t;\cdot)$ therefore as a strong solution to an evolution equation in L_2. Hence we have

$$\tilde{z}(t;\cdot) = \Phi(t)z(0;\cdot) - \int_0^t \Phi(t-s)\left[\int_{-r}^{\cdot} d\mu(\xi)\dot{\tilde{x}}^0(s) + \int_{-r}^{\cdot} d\zeta(\xi)\dot{u}(s)\right]ds.$$

By using (4.8) and integration by parts we obtain

$$(4.9)\qquad \tilde{z}(t;0) = \tilde{x}^1(0;\theta-t)x_{[-r,0]}(\theta-t) - \int_{-r}^{\theta} d\mu(\xi)\tilde{x}^0(t) - \int_{-r}^{\theta} d\zeta(\xi)u(t)$$
$$+ \int_{\max[-r,\theta-t]}^{\theta} \left(d\mu(\xi)\tilde{x}^0(t+\xi-\theta) + d\zeta(\xi)u(t+\xi-\theta)\right).$$

Now suppose $t > r$. Then it follows form (4.6) and (4.7) that

$$(4.10)\qquad \tilde{x}^1(t;\theta) = \int_{-r}^{\theta} \left(d\mu(\xi)\tilde{x}^0(t+\xi-\theta) + d\zeta(\xi)u(t+\xi-\theta)\right).$$

Hence (4.3) holds if $\psi \in \mathcal{D}(A^*)$ and $u \in U$.

Next suppose that the initial function $\left((\phi^0,\phi^1),\eta\right)$ of the equation (2.1) belongs to the subspace Q:

$$Q = \left\{\left((h(0),h(\cdot)),v(\cdot)\right) \in M_2 \times L_2 \,\middle|\, \dot{h},\dot{v} \in L_2 \text{ and } v(0) = 0\right\},$$

and take $\psi = M((\phi^0,\phi^1),\eta)$ as an initial function of (3.6). Then $\psi \in \mathcal{D}(A^*)$. Indeed

$$[A^*\psi]^1(\theta) = \int_{-r}^{\theta} d\mu(\xi)\dot{\phi}^1(\xi-\theta) + \int_{-r}^{\theta} d\zeta(\xi)\dot{\eta}(\xi-\theta) \in L_2.$$

For such a choice of the initial function, by (4.6) (4.7), and (4.9) we have

$$\tilde{x}^1(t:\theta) = \int_0^{\theta-t} \left(d\mu(\xi)\phi^1(t+\xi-\theta) + d\zeta(\xi)\eta(t+\xi-\theta)\right)$$

$$+ \int_{\theta-t}^{\theta} \left(d\mu(\xi)\tilde{x}^0(t+\xi-\theta) + d\zeta(\xi)u(t+\xi-\theta)\right),$$

$$\text{for} \quad -r < \theta-t < 0 \quad \text{and} \quad t > 0,$$

which may be written as

$$(4.11) \qquad \tilde{x}^1(t;\theta) = \int_0^{\theta} \left(d\mu(\xi)\tilde{x}^0(t+\xi-\theta) + d\zeta(\xi)u(t+\xi-\theta)\right),$$

when we take $\tilde{x}^0(\theta) = \phi^1(\theta)$, $u(\theta) = \eta(\theta)$ for $\theta \in [-r,0]$, (recall $\phi^1(0) = \phi^0 = \tilde{x}^0(0)$ and $\eta(0) = u(0) = 0$). It follows now from (4.4) that the locally absolutely continuous function on $[-r, \infty)$ defined to be $\phi^1(t)$, $z \in [-r,0]$ and $\tilde{x}^0(t)$, $t \in [0,\infty)$, coincides with the solution $x(t)$, $t > -r$ to (2.1) because of the uniqueness of solutions of (2.1). Hence from (4.10) and (4.11) we see that (4.2) holds if $(\phi,\eta) \in Q$ and $u \in U$.

Note that Q is dense in $M_2 \times L_2([-r,0]; \mathbb{R}^m)$ and $\mathcal{D}(A^*) \times U$ is dense in $M_2 \times L_2([0,t]; \mathbb{R}^m)$ for any $t > 0$. Hence by using the argument in [4, pg. 145], which is an application of extension by continuity arguments, we may show (4.2) holds for any initial function $(\phi,\eta) \in M_2 \times L_2$ and (4.3) holds for any initial function $\phi \in M_2$ and control $u \in L_2^{loc}([0,\infty); \mathbb{R}^m)$.

(Q.E.D.)

5. A Finite Time Quadratic Cost Control Problem

Let C be a $p \times N$ matrix and let the initial function $((\phi^0,\phi^1),\eta) \in M_2 \times L_2$ be given.

Consider the following control problem:

$$\text{(5.1)} \qquad \text{Minimize} \quad J([0,T];u) = \int_0^T (|Cx(t)|^2 + |u(t)|^2)dt$$

over $u(\cdot) \in L^2([0,T]; \mathbb{R}^m)$, subject to (2.1).

Define $\mathcal{C} \in \mathcal{L}(\mathbb{R}^p, M_2)$ by

$$\text{(5.2)} \qquad \mathcal{C} y = (C^T y, 0) \in M_2 \quad \text{for} \quad y \in \mathbb{R}^p.$$

Then by Theorem 4.1, (5.1) may be reformulated as

$$\text{(5.3)} \qquad \text{Minimize} \quad \int_0^T (|\mathcal{C}^* \tilde{x}(t)|^2 + |u(t)|^2)dt,$$

over $u(\cdot) \in L_2([0,T]; \mathbb{R}^m)$, where $\tilde{x}(t)$, $t \geq 0$ is given by (3.6) with initial function $\phi = M((\phi^0,\phi^1),\eta)$.

Hence applying Proposition 6.3 and Theorem 6.6 in [5] we obtain the following:

Theorem 5.1. The unique solution u^* to the control problem (5.1) is given by

$$\text{(5.4)} \qquad u^*(t) = - \mathcal{B}\mathcal{P}_T(t)\tilde{x}^*(t),$$

where $\tilde{x}^*(t)$ is the optimal trajectory corresponding to u^* of (3.6), and $P_T(t)$ is the solution to the Riccati equation

$$(5.5) \qquad \frac{d}{dt} \langle\langle P(t)\phi,\phi\rangle\rangle - 2\langle\langle AP(t)\phi,\phi\rangle\rangle + \langle BP(t)\phi, BP(t)\phi\rangle - \langle C^*\phi, C^*\phi\rangle,$$

$$\text{for all } \phi \in M_2 \text{ and } P(T) = 0.,$$

which is unique in the class of nonnegative definite, self-adjoint operators operators on M_2 satisfying (i) $\|AP(t)\phi\|$ is locally square integrable and (ii) $\langle P(t)\phi,\phi\rangle$ is locally absolutely continuous for all $\phi \in M_2$.

Moreover we have

$$(5.6) \qquad \langle P_T(0)M(\phi,\eta),M(\phi,\eta)\rangle = \inf J([0,T]:u).$$

From Theorem 4.1 $\tilde{x}^*(t) = M\big((x^*(t),x_t^*(\cdot)),u_t^*(\cdot)\big)$ where $x^*(\cdot)$ is the solution of (2.1) corresponding to u^* and hence

$$(5.7) \qquad u^*(t) = - BP_T(t)M\big(x^*(t),x_t^*(\cdot),u_t^*(\cdot)\big).$$

6. The Infinite Time Problem

In this section we consider the infinite time version of the control problem (5.1), i.e.,

$$(6.1) \qquad \text{Minimize} \int_0^\infty \left(|Cx(t)|^2 + |u(t)|^2\right) dt,$$

over $u \in L_2([0,\infty); R^m)$, subject to (2.1), given initial function

$(\phi,\eta) \in M_2 \times L_2$.

For any $K \in \mathcal{L}(M_2, \mathbb{R}^m)$ it is known [5, Lemma 6.4] that the equation

$$(6.2) \qquad \tilde{x}(t) = T^*(t)\phi + \int_0^t T'(t-s)\mathcal{B}'K\tilde{x}(s)ds,$$

has a unique continuous solution for each $\phi \in M_2$.

Define $\mathcal{U}(t) \in \mathcal{L}(M_2)$ as

$$(6.3) \qquad \mathcal{U}(t)\phi = \tilde{x}(t) \qquad \text{for each} \quad t \geq 0,$$

where $\tilde{x}(\cdot)$ is the solution to (6.2). Then it is easy to show that $\mathcal{U}(t)$, $t \geq 0$ satisfies the semigroup property. Hence $\mathcal{U}(t)$, $t \geq 0$ defines a strongly continuous semigroup on M_2.

Note that

$$(6.4) \qquad \frac{\partial}{\partial t}\mathcal{U}(t)\phi = \mathcal{U}(t)(\mathcal{A}' + \mathcal{B}'K)\phi \quad \text{in} \quad X',$$

for any $\phi \in M_2$. Hence the domain of the infinitesimal generator associated with $\mathcal{U}(t)$, $t \geq 0$ is given by $\{\phi \in M_2 | (\mathcal{A}' + \mathcal{B}'K)\phi \in M_2\}$, which is not equal to $\mathcal{D}(\mathcal{A}^*)$ in general.

Definition 6.1

(i) The pair $(\mathcal{A},\mathcal{B})$ is stabilizable if there exists an operator $K \in \mathcal{L}(M_2, \mathbb{R}^m)$ such that the semigroup $\mathcal{U}(t)$, $t \geq 0$, defined by (6.3) is exponentially stable, i.e.,

$$\|\mathcal{U}(t)\| \leq Me^{-\omega t}, \; t \geq 0$$

for some positive constants M and ω

(ii) The pair (A,C) is detectable if there exists an operator $F \in L(\mathbb{R}^p, M_2)$ such that $A^* + FC^*$ generates an exponentially stable semigroup on M_2.

It should be noted that by the theorem due to Datko [2] this definition of stabilizability and detectability is equivalent to that defined in [11, Definition 7.1].

The following is the solution to the control problem (6.1)

Theorem 6.2. Consider the equation in P

$$(6.5) \qquad 2 \langle\langle AP\phi,\phi\rangle\rangle - \langle BP\phi,BP\phi\rangle + \langle C^*\phi,C^*\phi\rangle = 0, \qquad \text{for all } \phi \in M_2,$$

within a class of nonnegative definite, self-adjoint operators in $L(M_2)$ such that $AP \in L(M_2)$. Then

(i) If (A,B) is stabilizable, then (6.5) has a solution

(ii) If (A,B) is detectable, then (6.5) has at most one solution, and if P is the solution, then $A' - B'BP$ generates an exponentially stable semigroup.

(iii) If (A,B) is stabilizable and (A,C) is detectable then

$$P_T(0) \to P(\text{strongly}), \quad \text{as } T \to \infty,$$

where P is a unique solution to (6.5), and $P_T(\cdot)$ is the unique solution to (5.5) for each $T > 0$, and the optimal solution to (6.1) is given by

$$u^*(t) = -\mathcal{B}P\tilde{x}^*(t),$$

where $\tilde{x}^*(\cdot)$ satifies

$$\tilde{x}(t) = T^*(t)\psi - \int_0^t T'(t-s)\mathcal{B}'K\,\tilde{x}(s)ds,$$

$$\psi = M((\phi^0,\phi^1),\eta).$$

From Theorem 4.1 we have $\tilde{x}^*(t) = M((x^*(t),x_t^*(\cdot)),u_t^*(\cdot))$, where $x^*(\cdot)$ is the solution corresponding to u^*.

Moreover,

$$\langle\langle PM(\phi,\eta),M(\phi,\eta)\rangle\rangle = \inf J([0,\infty):u).$$

7. Proof of Theorem 6.2.

Under the assumption of stabilizability there exists a K such that the corresponding semigroup $\mathcal{U}(t)$, $t \geq$ is exponentially stable. Then

$$\text{(7.1)}\qquad \inf\{J([0,\infty);u)\,|\,u \in L_2([0,\infty);\mathbb{R}^m)\}$$

$$\leq (\|C\|^2 + \|K\|^2)\int_0^\infty \|\mathcal{U}(t)\psi\|^2dt$$

$$\leq \beta\|\psi\|^2 \quad \text{for some positive constant } \beta.$$

It follows from the relation (5.6), (note that (5.6) holds for an arbitrary initial function $\psi \in M_2$) that $P_T(0)$, $T > 0$ forms a monotone nondecreaing sequence of self-adjoint operators on M_2, uniformly bounded above. Hence by Theorem 2 [12, pp. 304] $P_T(0)$ converges strongly to some self-adjoint operator P satisfying $P \leq \beta I$.

However since $\mathcal{B}$ is not in $\mathcal{L}(M_2,\mathbb{R}^m)$, this may not imply the convergence of $\mathcal{B}\mathcal{P}_T(0)\ \phi$ for any $\phi \in M_2$. Hence we need further investigation.

<u>Lemma 7.1.</u> $\mathcal{A}\mathcal{P}_T(0)\phi \to \mathcal{A}\mathcal{P}\phi$ (strongly) in M_2 for any $\phi \in M_2$.

<u>Proof:</u> By using standard arguments in [6] and [1, Chapter 3] we may show that

$$\mathcal{P}_T(0)\phi = \int_0^T T(s)\, \mathcal{C}\mathcal{C}^* \tilde{x}_T^*(s)ds,$$

where $\tilde{x}_T^*(\cdot)$ is the optimal trajectory corrresponding to u^* for each $T > 0$. Since $\mathcal{C}\mathcal{C}^*\tilde{x}_T^*(t) = (C^T C\tilde{x}_T^{0^*}, 0) \in M_2$ for each $t > 0$, it may be written as

$$[\mathcal{P}_T(0)\phi](\theta) = \int_0^T X(s+\theta)C^T C\tilde{x}_T^{0^*}(s)ds, \qquad \theta \in [-r,0],$$

where $X(t)$, $t > -r$ is the fundamental matrix solution of the equation (2.1): i.e.,

$$X(t) = I + \int_0^t L^T X_s ds, \tag{7.2}$$

$$X(0) = I \text{ and } X(\theta) = 0, \qquad \theta \in [-r,0).$$

Since $\mathcal{P}_T(0)\phi \in \mathcal{D}(\mathcal{A})$ for any $\phi \in M_2$, we have from Lemma 3.1

$$[\mathcal{A}P_T(0)\phi]^1(\theta) = \frac{d}{d\theta} \int_0^T X(s+\theta)C^T C\tilde{x}_T^{0^*}(s)ds,$$

$$= \int_0^T dX(s+\theta)C^T C\tilde{x}_T^{0^*}(s), \qquad \theta \in [-r,0].$$

Using (7.1) we obtain

$$(7.3) \qquad [AP_T(0)]^1(\theta) = A_0^T \int_0^T X(s+\theta)C^T C\tilde{x}_T^{0^*}(s)ds$$

$$+ \int_0^T \int_{-r}^0 d\mu^T(\xi)X(s+\theta+\xi)C^T C\tilde{x}_T^{0^*}(s)ds + C^T C\tilde{x}_T^{0^*}(-\theta),$$

where it should be noticed that $-2r \leq \theta + \xi \leq 0$.

Now we view our problem as that defined on the extended interval $[-2r,0]$. Let $P_T^e(\cdot)$ be a solution of the Riccati equation in $\mathbb{R}^N \times L_2([-2r,0];\mathbb{R}^N)$

$$\frac{d}{dt}\langle P^e(t)\phi,\phi\rangle_e = -2\langle A^e P^e(t)\phi,\phi\rangle_e$$
$$+ \langle B^e P^e(t)\phi, B^e P^e(t)\phi\rangle - \langle C^{e^*}\phi, C^{e^*}\phi\rangle$$

$P^e(T) = 0$ for all $\phi = (\phi^0,\phi^1) \in \mathbb{R}^N \times L_2([-2r,0];\ \mathbb{R}^N)$,

where $\langle\cdot,\cdot\rangle_e$ denotes the natural inner product in $\mathbb{R}^N \times L_2([-2r,0];\mathbb{R}^N)$, and if $\pi \in (L_2([-2r,0];\ \mathbb{R}^N), L_2([-r,0];\mathbb{R}^N))$ is defined by $\pi\phi^1 = \phi^1\chi_{[-r,0]}$ for $\phi^1 \in L_2([-2r,0];\ \mathbb{R}^N)$, then

$$\mathcal{D}(A^e) = \{\phi \in \mathbb{R}^N \times L_2([-2r,0];\ \mathbb{R}^N)\ \dot{\phi}^1 \in L_2 \text{ and } \phi^0 = \phi^1(0)\},$$

$$A^e\phi = (L^T\pi\phi^1,\dot{\phi}^1) \in \mathbb{R}^N \times L_2,$$

$$B^e\phi = B(\phi^0,\ \pi\phi^1),$$

$$C^{e*}\phi = C\phi^0,$$

for $\phi = (\phi^0,\phi^1) \in \mathbb{R}^N \times L_2([-2r,0];\ \mathbb{R}^N)$.

Let us consider the corresponding control problem:

$$\text{Minimize} \quad J^e([0,T];u) = \int_0^T \left(|C^{e*}\tilde{x}^e(t)|^2 + |u(t)|^2\right)dt,$$

over $u \in L_2([0,T];\mathbb{R}^m)$, subject to

$$\tilde{x}^e(t) = T^e(t)^*\phi + \int_0^t T^e(t-s)'B^{e'}u(s)ds,$$

$$\tilde{x}^e(0) = \phi \in \mathbb{R}^N \times L_2([-2r,0];\ \mathbb{R}^N),$$

where $T^e(t)$, $t \geq 0$ is the semigroup generated by A^e. Note that by Theorem 4.1, for $u \in L_2^{loc}([0,\infty);\ \mathbb{R}^m)$ such that $u(t) = 0$ for $t \in [0,2r]$, $\tilde{x}^e(\cdot)$ satisfies

$$\tilde{x}^e(2r) = M^e\left((\tilde{x}^{0^e}(2r),\tilde{x}^{0^e}_{2r}(\cdot)),0\right),$$

and

$$\tilde{x}^e(t) = M^e\left((\tilde{x}^{0^e}(t),\ \tilde{x}^{0^e}_t(\cdot)),u_t(\cdot)\right),$$

where

$$M^e\left((\phi^0,\phi^1),\eta\right) = \left(\phi^0,m^e(\cdot)\right),$$

$$m^e(\theta) = \begin{cases} \int_{-r}^{\theta} d\mu(\xi)\phi^1(\xi-\theta) + \int_{-r}^{\theta} d\zeta(\xi)\eta(\xi-\theta), & \theta \in [-r,0] \\ 0, & \text{otherwise.} \end{cases}$$

Now let $u(\cdot)$ be chosen as follows

$$u(t) = \begin{cases} 0 & t \in [0,2r], \\ K\left(\tilde{x}^{0^e}(t),\ \pi\tilde{x}^{1^e}(t)\right), & t > 2r. \end{cases}$$

Then it is easily verified that

$$\ll P_T^e(0)\phi,\phi \gg \, = \inf J^e < \int_0^{2r} |C\tilde{x}^{0^e}(t)|^2 \, dt + \beta \|\tilde{x}^e(2r)\|^2.$$

Hence $P_T^e(0)$ is uniformly bounded above and so $P_T^e(0)$ converges strongly in $\mathbb{R}^N \times L_2([-2r,0];\ \mathbb{R}^N)$.

Now since $[P_T(0)\ (\phi^0\pi\phi^1)]^1 = \pi[P_T^e(0)(\phi^0,\phi^1\chi_{[-r,0]})]^1$, it follows from (7.3) that

$$(7.4)\ [AP_T(0)\phi]^1(\theta) = A_0^T[P_T^e(0)\tilde{\phi}]^1(\theta) + \int_{-r}^{0} d\mu^T(\xi)[P_T^e(0)\tilde{\phi}](\theta+\xi) + C^T C\tilde{x}_T^{0^*}(-\theta),$$

where

$$\tilde{\phi}^1(\theta) = \phi^1\chi_{[-r,0]}(\theta) \quad \text{for} \quad \theta \in [-2r,0] \quad \text{and} \quad \tilde{\phi}^0 = \phi^0.$$

By using the same arguments in [9, Theorem 4.1, p. 143] and [6, Proposition 3.1] we may show that $C^*\tilde{x}_T^*(-\theta)$ converges strongly in L_2 as $T \to \infty$. It now follows from, (7.4) that $[AP_T(0)\phi]^1(\cdot)$ converges strongly in L_2, which when combined with the fact that $\mathcal{D}(d/d\theta) \subset \mathcal{D}(L^T)$ and $[AP\phi_T(0)\phi]^0 = L^T P_T(0)\phi$, implies $[AP_T(0)\ \phi\ ]^0$ converges in $\mathbb{R}^N$. Hence we may conclude that $AP_T(0)\phi$ converges strongly in M_2 for any $\phi \in M_2$. Since A is closed and $P_T(0)\phi \to P\phi$ strongly, this implies $A P_T(0)\phi$ converges strongly to $AP\phi$ for any $\phi \in M_2$. (Q.E.D.)

Since $\mathcal{D}(A) \subset \mathcal{D}(B)$ it follows from Lemma 7.1 that $BP_T(0)\phi \to BP\phi$ in $\mathbb{R}^m$ for any $\phi \in M_2$. Then it is easy to show that P satisfies (6.5) [6].

To prove the uniqueness under the assumption of detectability we need the following lemma.

Lemma 7.2. Suppose there exists a solution P to (6.5) and (A,C) is detectable. Then $A' - B'BP$ generates an exponentially stable semigroup on M_2.

Proof: Since (A,C) is detectable there exists an operator $F \in L(\mathbb{R}^P, M_2)$ such that $A^* + CF^*$ generates an exponentially stable semigroup $S(t)$, $t > 0$ on M_2, i.e.,

$$\|S(t)\| < \kappa e^{-\omega t}, \ t > 0 \qquad \text{for some positive constants } \kappa \text{ and } \omega.$$

Suppose $U(t)$, $t > 0$ is the semigroup generated by $A' - B'B\phi$, then we may show that

$$(7.5) \qquad U(t)\phi = S^*(t)\phi - \int_0^t S^*(t-s)FC^*U(s)\phi ds - \int_0^t S'(t-s)B'BPU(s)\phi ds,$$

for any $\phi \in M_2$ since

$$A' - B'BP = (A' + FC^*) - (FC^* + B'BP) \quad \text{in } X',$$

and $A + CF^*$ is associated with the functional differential equation in $\mathbb{R}^N$

$$\frac{d}{dx} x(t) = (L^T + CF^*)x_t.$$

Now suppose $z(t) = \mathcal{U}(t)\,\phi$. Then we show that

$$(7.6) \qquad \int_0^\infty |\mathcal{C}^* z(t)|^2 \, dt < \infty \quad \text{and} \quad \int_0^\infty |\mathcal{B}\mathcal{P} z(t)|^2 dt < \infty,$$

for any $\phi \in M_2$.

From the assumption

$$2 \langle\!\langle \mathcal{A}\mathcal{P}z(t), z(t)\rangle\!\rangle - \langle \mathcal{B}\mathcal{P}z(t), \mathcal{B}\mathcal{P}z(t)\rangle + \langle \mathcal{C}^* z(t), \mathcal{C}^* z(t)\rangle = 0,$$

for each $t > 0$, it follows from [5, Theorem 6.5] that

$$\frac{d}{dt} \langle\!\langle \mathcal{P}z(t), z(t)\rangle\!\rangle = 2\langle\!\langle \mathcal{A}\mathcal{P}(t)z(t), z(t)\rangle\!\rangle - 2\langle \mathcal{B}\mathcal{P}z(t), \mathcal{B}\mathcal{P}z(t)\rangle.$$

Hence we have

$$\frac{d}{dt} \langle\!\langle \mathcal{P}z(t), z(t)\rangle\!\rangle = \langle \mathcal{B}\mathcal{P}z(t), \mathcal{B}\mathcal{P}\, z(t)\rangle + \langle \mathcal{C}^* z(t), \mathcal{C}^* z(t)\rangle = 0,$$

and therefore

$$(7.7) \qquad \langle\!\langle \mathcal{P}z(t), z(t)\rangle\!\rangle = \int_0^t |\mathcal{B}\mathcal{P}z(s)|^2 ds + \int_0^t |\mathcal{C}^* z(s)|^2 ds = \langle\!\langle \mathcal{P}\phi, \phi\rangle\!\rangle,$$

for each $t > 0$, from which (7.6) holds.

Let us define

$$z_3(t) = \int_0^t S'(t-s)\mathcal{B}'\mathcal{B}z(s)ds,$$

and

$$u(t) = -\ \mathcal{B}\mathcal{P}z(t),\ t \geq 0.$$

Then it follows from Theorem 4.1 that

$$z_3(t) = M\big(x(t), x_t(\cdot), u_t(\cdot)\big),$$

where $x(\cdot)$ is the solution to (2.1) with initial function $(\phi,\eta) = (0,0)$. (The operator M and $x(\cdot)$ are associated with the perturbed system $\dot{x}(t) = (\mathcal{L}^T + C\mathcal{F}^*)^T x_t + Bu_t$ of (2.1).) But we have

$$\big(x(t), x_t(\cdot)\big) = \int_0^t S(t-s)(Bu_s, 0)ds,$$

where

$$\left(\int_0^\infty |Bu_s|^2 ds\right)^{1/2} \leq \left(\int_{-r}^0 |d\zeta(\theta)|\right)^{1/2} \left(\int_{-r}^0 |d\zeta(\theta)| \quad \int_0^\infty |u(s+\theta)|^2 ds\right)^{1/2}$$

$$+ \ |B_0| \left(\int_0^\infty |u(s)|^2 ds\right)^{1/2},$$

$$\leq \left(|B_0| + \int_{-r}^0 |d\zeta(\theta)|\right) \left(\int_0^a |u(s)|^2 ds\right)^{1/2}.$$

Hence

$$\left(\int_0^\infty \|(x(t), x_t)\|^2_{M_2}\ dt\right)^{1/2} \leq \frac{\kappa}{\omega}\left(|B_0| + \int_{-r}^0 |d\zeta(\theta)|\right)\left(\int_0^\infty |u(s)|^2 ds\right)^{1/2},$$

using Young's inequality. Since M is continuous from $M_2 \times L_2$ into M_2, we have

$$\left(\int_0^\infty \|z_3(t)\|^2 dt\right)^{1/2} \leq \|M\| \frac{\kappa}{\omega}\left(|B_0| + \int_0^\infty |d\zeta(\theta)|\right)\left(\int_0^\infty |u(s)|^2 ds\right)^{1/2}. \tag{7.8}$$

Now we have from (7.5)

$$\|z(t)\| \leq \|S^*(t)\phi\| + \int_0^t \|S(t-s)\| \; \|\mathcal{F}\| \, |\mathcal{C}^* z(s)| \, ds + \|z_3(t)\|,$$

and therefore

$$\left(\int_0^\infty \|z(t)\|^2 dt\right)^{1/2} \leq \left(\frac{\kappa^2}{2\omega}\right)^{1/2} \|\phi\| + \frac{\kappa}{\omega} \|\mathcal{F}\| \left(\int_0^\infty |\mathcal{C}^* z(s)^2| ds\right)^{1/2} + \left(\int_0^\infty \|z_3(t)\|^2 dt\right)^{1/2}$$

Hence we obtain, from (7.6) and (7.8)

$$\int_0^\infty \|z(t)\|^2 dt < \infty, \text{ for any } \phi \in M_2.$$

Now by Datko's theorem [2], we may conclude that $\mathcal{U}(t)$, $t \geq 0$ is exponentially stable. (Q.E.D.)

We now turn to prove the assertion (ii) of Theorem 6.2. Suppose there exists two solution $\mathcal{P}_1$ and $\mathcal{P}_2$ to (6.5). Then we have, from (7.7)

$$\int_0^\infty \left(|\mathcal{B}\mathcal{P}_i z_i(s)|^2 + |\mathcal{C}^* z_i(s)|^2 \right) ds = \langle \mathcal{P}_i \phi, \phi \rangle, \tag{7.9}$$

for any $\phi \in M_2$ and $i = 1,2$, where $z_i(\cdot)$ satisfies

$$z_i(t) = T^*(t)\phi - \int_0^t T'(t-s)\mathcal{B}'\mathcal{B}\mathcal{P}_i z_i(s) ds.$$

Let us define

$$v(t) = -\mathcal{B}\mathcal{P}_2 z_2(t) + \mathcal{B}\mathcal{P}_1 z_2(t), \qquad t \geq 0.$$

Then

$$\langle\!\langle P_2\phi,\phi\rangle\!\rangle = \int_0^\infty \big(|v(t)|^2 - 2\langle v(t),BP_1z_2(t)\rangle + \langle BP_1z_2(t),BP_1z_2(t)\rangle + \langle C^*z_2(t),C^*z_2(t)\rangle\big)dt.$$

But it follows from [5, Lemma 6.5] that

$$\frac{d}{dt}\langle\!\langle P_1z_2(t),z_2(t)\rangle\!\rangle = 2\langle\!\langle AP_1z_2(t),z_2(t)\rangle\!\rangle - 2\langle BP_1z_2(t),BP_2z_2(t)\rangle,$$

$$= \langle BP_1z_2(t),BP_1z_2(t)\rangle - \langle C^*z_2(t),C^*z_2(t)\rangle - 2\langle BP_1z_2(t),BP_2z_2(t)\rangle,$$

$$= -\langle BP_1z_2(t),BP_1z_2(t)\rangle - \langle C^*z_2(t),C^*z_2(t)\rangle + 2\langle BP_1z_2(t),v(t)\rangle.$$

Hence we obtain

$$\langle\!\langle P_2\phi,\phi\rangle\!\rangle = \int_0^a |v(t)|^2dt + \int_0^\infty -\frac{d}{dt}\langle\!\langle P_1z_2(t),z_2(t)\rangle\!\rangle\,dt,$$

$$= \int_0^\infty |v(t)|^2dt + \langle\!\langle P_1\phi,\phi\rangle\!\rangle - \lim_{t\to\infty}\langle\!\langle P_1z_2(t),z_2(t)\rangle\!\rangle,$$

$$= \langle\!\langle P_1\phi,\phi\rangle\!\rangle + \int_0^\infty |v(t)|^2dt,$$

for any $\phi \in M_2$, where we need the fact that $\|z_2(t)\| \to 0$ as $t \to \infty$ for any $\phi \in M_2$.

This implies $P_1 \leq P_2$. Likewise we may prove that $P_1 \geq P_2$, whence, by the property of partial orderings $P_1 \equiv P_2$.

Finally we prove the last statement of Theorem 6.2. Consider any control $u \in L_2^{loc}([0,\infty); \mathbb{R}^m)$. Then for each $t \geq 0$ and $\phi \in M_2$,

$$\langle \mathcal{P}_T(t)\phi,\phi\rangle \leq \int_0^t (|u(s)|^2 + |C^*\tilde{x}(s)|^2)ds,$$

where $\tilde{x}(\cdot)$ is given by (3.6). Since $\mathcal{P}_T(0) \to \mathcal{P}$ strongly this implies

$$\langle \mathcal{P}\phi,\phi\rangle = \int_0^\infty (|C^*\tilde{x}(s)|^2 + |u(s)|^2)ds.$$

Hence it follows from the uniqueness of the optimal control and (7.9) that

$$u^*(t) = -\mathcal{B}\mathcal{P}\, z(s),$$

is the optimal solution to (6.1).

8. Filter Stability

In this section we discuss the stability of a filter equation for the stochastic delay system

$$x(t) = \phi(0) + \int_0^t L^T x_s ds + G\, W(t), \qquad x_0 = \phi, \tag{8.1}$$

$$y(t) = \int_0^t H x_s ds + F^{1/2} V(t).$$

Here the observation process $y(\cdot)$ takes values in $\mathbb{R}^P$. The linear map $H: L_2([-r,0];\mathbb{R}^N) \to \mathbb{R}^P$ is defined by

$$H\phi = \int_{-r}^0 d\gamma(\theta)\phi(\theta), \tag{8.2}$$

where γ is a matrix valued function of bounded variation in $(-r,0)$.

$W(t), V(t)$ are independent standard Wiener processes in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. The matrix F is positive definite and symmetric. The initial function ϕ is assumed to be deterministic and continuous.

Define the abstract state $z(t) \varepsilon M_2$ by

$$z(t) = \left(x(t), x_t(\cdot)\right) \varepsilon M_2 \qquad \text{for each} \quad t \geq 0.$$

Then it follows from [5] that $z(t)$ is $C([-r,0];\mathbb{R}^N)$ - valued, Gaussian and continuous w.p.1, and satisfies the stochastic evolution equation

(8.4)

$$\langle\langle z(t),\psi\rangle\rangle = \langle\langle z(0)\psi\rangle\rangle + \int_0^t \langle\langle z(s), A^*\psi\rangle\rangle\, ds + \langle W(t), \tilde{G}^*\psi\rangle \qquad \text{for all} \quad \psi \varepsilon \mathcal{D}(A^*),$$

where

$$z(0) = \left(\phi(0), \phi(\cdot)\right) \varepsilon M_2,$$

and

$$\tilde{G} \varepsilon \mathcal{L}(\mathbb{R}^m, M_2) \quad \text{is defined by,}$$

(8.5) $$\tilde{G}u = (Gu, 0) \varepsilon M_2 \qquad \text{for} \quad u \varepsilon \mathbb{R}^m.$$

Let us denote by $\hat{z}(t)$ the conditional expectation $E[z(t)|y(s),$ $0 \leq s \leq t]$. Then $\hat{z}(t)$ is given by

(8.6) $$\hat{z}(t) = T(t)z(0) + \int_0^t T(t-s)[\mathcal{H}\mathcal{P}(s)]^* F^{-1} dI(s),$$

where $\mathcal{H}: M_2 \to \mathbb{R}^p$ is defined by

(8.7) $$\mathcal{H}\phi = H\phi^1 \quad \text{for} \quad \phi = (\phi^0,\phi^1) \in M_2,$$

and the innovation process I(t) is defined by

(8.8) $$I(t) = y(t) - \int_0^t \mathcal{H}\hat{z}(s)ds, \quad \text{for each } t > 0.$$

The error covariance operator $\mathcal{P}(t)$, $t > 0$, defined by

(8.9) $$\langle \mathcal{P}(t)\phi,\phi\rangle = E[\langle\langle z(t) - \hat{z}(t),\phi\rangle\rangle \langle\langle z(t)-\hat{z}(t),\phi\rangle\rangle],$$

satisfies a Riccati equation:

(8.10) $$\frac{d}{dt}\langle\langle \mathcal{P}(t)\phi,\phi\rangle\rangle = 2\langle\langle \mathcal{A}\mathcal{P}(t)\phi,\phi\rangle\rangle - \langle \mathcal{H}\mathcal{P}(t)\phi,\mathcal{H}\mathcal{P}(t)\phi\rangle_{F^{-1}} + \langle \tilde{G}^*\phi,\tilde{G}^*\phi\rangle,$$
$$\text{for all } \phi \in M_2 \quad \text{and} \quad \mathcal{P}(0) = 0,$$

where $\langle\cdot,\cdot\rangle_{F^{-1}}: \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is defined by

$$\langle y_1,y_2\rangle_{F^{-1}} = y_1^T F^{-1} y_2, \quad \text{for } y_1,y_2 \in \mathbb{R}^p,$$

and is the unique solution of (8.10) within the class of nonnegative definite, self-adjoint operators on M_2 satisfying (i) $\|\mathcal{A}\mathcal{P}(t)\phi\|$ is locally square integrable and (ii) $\langle\langle \mathcal{P}(t)\phi,\phi\rangle\rangle$ is locally absolutely continuous for any $\phi \in M_2$.

The the next theorem follows from Theorem 6.2.

<u>Theorem 8.1.</u> Suppose $(\mathcal{A},\mathcal{H})$ is stabilizable and $(\mathcal{A}, \tilde{G})$ is detectable. Then the optimal filter equation defined by (8.6) and (8.10) is stable in the following sense.

(i) The error covariance operator $\mathcal{P}(t)$ converges strongly to $\mathcal{P}$ which satisfies

$$2\langle AP\phi,\phi\rangle - \langle HP\phi,HP\phi\rangle_{F^{-1}} + \langle\tilde{G}^*\phi,\tilde{G}^*\phi\rangle = 0, \qquad \text{for all} \quad \phi \in M_2.$$

(ii) The closed loop operator $A - [HP]^*F^{-1}H$ generates an exponentially stable semigroup on M_2.

Corollary 8.2. The stationary filter equation is given by

$$\hat{z}(t) = U(t)z(0) + \int_0^t U(t-s)F^{-1}dy(s),$$

where $U(t)$, $t \geq 0$ is the semigroup on M_2 generated by $A - [HP]^*F^{-1}H$.

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